Summer Assignment for AP Calculus AB Students
Who Have Signed Up For Advanced Calculus II

In an effort to bridge part of the gap that will exist between you and those who are in Advanced Calculus II after having taken AP Calculus BC last year, I am asking that you learn the five sections I've included here. These **will be** assumed knowledge in our course. These topics will be revisited during the course, but will be treated as if you have seen the material before and understand it. Some of this material will be on our initial review quiz and review test at the beginning of the year.

Each of the sections contains the notes I gave in class, worked out examples, an "assignment" for further practice, and the answers for that assignment. As you work through these sections, please do not hesitate to e-mail me at my school e-mail if you have questions. I will be checking my e-mail at least once a day. Since this is the first summer for this assignment, please feel free to point out any typos you may find. ☺

The sections you will be learning are:

Best of luck, enjoy your summer, and I shall see you in August.

Mr. Field

<u>Euler's Method</u>

<u>(pronounced Oiler's)</u>

<u>Goal:</u> To find an approximate value of a function (with the calculator) using a differential equation

<u>Need:</u> a starting point and an initial slope

We will use increments of Δx to come up with successive approximations using these rules:

1. $x_n = x_{n-1} + \Delta x$ [Each *x* is equal to the previous *x* plus the change in *x*.]
2. $y_n = y_{n-1} + \Delta x\left(y'_{n-1}\right)$ [Each *y* is equal to the previous *y* plus the change in *x* multiplied by the previous slope.] Look at this formula and think about it. It makes sense. If you subtract y_{n-1} from both sides, the equation looks **<u>very much</u>** like our old friend point-slope: $y - y_1 = m(x - x_1)$.

Probably the best thing about Euler's Method is you will never have to guess when to use it. All questions dealing with Euler's method will begin, "Use Euler's Method to ……" and will also give you all the information you need. There is no hunting for information required.

<u>Example:</u> Use Euler's Method, with $\Delta x = 0.2$ to estimate $y(1)$ if $y' = y - 2$ and $y(0) = 4$.

To do this, I find that the best thing to do is make the following table:

n	x_n	$y_n = y_{n-1} + \Delta x\left(y'_{n-1}\right)$	$y' = y - 2$
0 (initial condition)	0	4	4 – 2 = 2
1	0.2 (initial *x* + change)	4 + 0.2 (2) = 4.4 Previous *y* + change(previous slope)	4.4 – 2 = 2.4
2	0.4	4.4 + 0.2 (2.4) = 4.88	4.88 – 2 = 2.88
3	0.6	4.88 + 0.2 (2.88) = 5.456	5.346 – 2 = 3.456
4	0.8	5.456 + 0.2 (3.456) = 6.1472	6.1472 – 2 = 4.1472
5	1.0	6.1472 + 0.2 (4.1472) = 6.97664	

Our approximation for $y(1) = 6.97664$.

Follow-up Question #1: Is this approximation too high or too low ?

As with all approximations we make in calculus, we are interested in knowing whether the approximation we obtained is too high or too low. The quickest way to answer this is to look at the values of the first derivative in our table. We notice that the values of the derivative are not only increasing, but also increasing at an **increasing rate.** This means that the derivative of the derivative (the 2^{nd} derivative) is positive. This implies that the function is monotonically concave up.

Q: What is Euler's Method doing?
A: Using tangent line approximations to estimate the value of a function.

Q: If a function is concave up, where will any tangent line drawn to the curve be?
A: Below the curve.

Therefore our estimation is too low.

Further Proof: We could actually solve the differential equation provided in the problem.

$$\frac{dy}{dx} = y - 2$$

$$\frac{dy}{y-2} = dx$$

$$\int \frac{dy}{y-2} = \int dx$$

$$\ln|y-2| = x + c$$

$$e^{\ln|y-2|} = e^{x+c}$$

$$y - 2 = Ce^{x}$$

$$4 - 2 = Ce^{0} \therefore C = 2$$

$$y = 2e^{x} + 2$$

$$y(1) = 2e^{1} + 2 \approx 7.4365$$

As shown earlier by looking at the concavity of the function, our approximation is too low.

Practice: In the following questions, use Euler's Method to approximate y (1) using the given conditions: You may use your calculator.

1) $\frac{dy}{dx} = y$, $y(0) = 1$, step size = 0.5

2) $\frac{dy}{dx} = y - x$, $y(0) = 2$, step size = 0.25

3) $\frac{dy}{dx} = 4x^3$, $y(0) = 0$, step size = 0.2

Answers: 1) 2.25 2) 4.44140625 3) 0.64

Integration by Parts

Goal: To anti-derive a product of two "unrelated" functions.

With u-substitution, we learned how to anti-derive products/quotients of two functions in which one of the functions was some constant multiple of the derivative of the other; the two functions were related to each other. In order to anti-derive a product in which no such relationship exists, we use integration by parts.

Suppose $f(x) = u \cdot v$ where u and v are two unrelated functions. Taking the derivative of both sides gives us: $f'(x) = (u \cdot v)' = uv' + vu'$ (Product Rule)

$(u \cdot v)' = uv' + vu'$

$(u \cdot v)' = udv + vdu$ (Simply rewriting derivative notation as differential notation.)

Integrating both sides gives us:

$$\int (u \cdot v)' = \int (udv + vdu)$$

$$\int (u \cdot v)' = \int udv + \int vdu$$

$$uv = \int udv + \int vdu$$

Subtracting $\int vdu$ from both sides gives $uv - \int vdu = \int udv$. Then rearranging gives us

$$\boxed{\int udv = uv - \int vdu}$$

Don't worry -- I don't expect you to know how to do it yet. The key to being successful is to correctly determine which of the two functions is *u*. In order to do this we use the acronym **LIPET.** LIPET stands for the different types of functions we study in mathematics.

L = logarithmic functions
I = inverse trigonometric functions
P = polynomial funycions
E = exponential functions
T = trigonometric functions

u will be the first type of function from LIPET we encounter in the integrand. *dv* will be the remainder of the integrand.

Let's see how this works.

Example 1: $\int x\cos x dx$

First of all, notice that *x* and cos*x* are unrelated fucntions and we can not therefore use *u*-substitution.
Secondly, we have a polynomial function (*x*) and a trigonometric function (cos*x*). Therefore, according to LIPET, we will let *u* = *x*, and *dv* = cos*xdx*.
Now, we will need to derive *u* to get *du*, and anti-derive *dv* to get *v*.

$u = x$	$dv = \cos x dx$
$du = dx$	$v = \sin x$

Now that we have all four pieces of the puzzle, we simply substitute the piees into the integration by parts formula: $\int udv = uv - \int vdu$

This gives us $\int x\cos x dx = x\sin x - \int \sin x dx$. The beauty of this is that we are now left with an integral that we know how to anti-derive.

So … $\int x\cos x dx = x\sin x - \int \sin x dx = x\sin x + \cos x + c$. You can check our answer by finding the derivative of our answer and seeing that it is equal to the original problem. ☺

Example 2: $\int x^4 \ln x dx$

Using LIPET, we will let $u = \ln x$ and $dv = x^4 dx$. We will need to derive *u* to get *du* and anti-derive *dv* to get *v*.

$u = \ln x$	$dv = x^4 dx$
$du = \frac{1}{x} dx$	$v = \frac{1}{5} x^5$

Using the formula for integration by parts:

$$\int x^4 \ln x dx = \frac{1}{5} x^5 \ln x - \int \frac{1}{5} x^5 \cdot \frac{1}{x} dx$$

$$= \frac{1}{5} x^5 \ln x - \int \frac{1}{5} x^4 dx = \frac{1}{5} x^5 \ln x - \frac{1}{25} x^5 + c$$

Example 3: $\int x e^{2x} dx$

Using LIPET, we will let $u = x$ and $dv = e^{2x} dx$. We will need to derive *u* to get *du* and anti-derive *dv* to get *v*.

$u = x$	$dv = e^{2x} dx$
$du = dx$	$v = \frac{1}{2} e^{2x}$

Using the formula for integration by parts:

$$\int x e^{2x} dx = \frac{1}{2} x e^{2x} - \int \frac{1}{2} e^{2x} dx$$

$$= \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + c$$

Example 4: $\int \ln x dx$

Using LIPET, we will let $u = \ln x$ and $dv = dx$. We will need to derive *u* to get *du* and anti-derive *dv* to get *v*.

$u = \ln x$	$dv = dx$
$du = \frac{1}{x} dx$	$v = x$

Using the formula for integration by parts:

$$\int \ln x dx = x \ln x - \int x \cdot \frac{1}{x} dx$$

$$= x \ln x - \int dx = x \ln x - x + c$$

Example 5: $\int x^4 \cos x dx$

We will start this problem the usual way, and see what happens.
Using LIPET, we will let $u = x^4$ and $dv = \cos x dx$. We will need to derive *u* to get *du* and anti-derive *dv* to get *v*.

$u = x^4$	$dv = \cos x dx$
$du = 4x^3 dx$	$v = \sin x$

Using the formula for integration by parts:

$$\int x^4 \cos x dx = x^4 \sin x - \int 4x^3 \sin x dx$$

And now we have a problem. We will have to use integration by parts again to evaluate $\int 4x^3 \sin x dx$. As it turns out we would have to use integration by parts a third time and a fourth time to get the final answer! Although possible, this approach is not very practical; lengthy **<u>and</u>** great chance of messing up the signs in the final answer. This gives way to a special case of integration by parts known as **tabular integration**. Tabular integration is VERY useful when the integrand is of the form $\int x^n \sin x dx$, $\int x^n \cos x dx$, or $\int x^n e^x dx$.

In order to do tabular integration, we will set up a two-column table with *u* in the first column and *dv* in the second column.

<u>u</u>	<u>dv</u>
x^4	$\cos x$

We will then **<u>derive</u>** *u* until we reach 0, and **<u>anti-derive</u>** *dv* until the table is complete.

<u>u</u>	<u>dv</u>
x^4	$\cos x$
$4x^3$	$\sin x$
$12x^2$	$-\cos x$
$24x$	$-\sin x$
24	$\cos x$
0	$\sin x$

We now draw diagonal arrows connecting elements of the table along with alternating signs as we go from arrow to arrow **<u>always</u>** starting with positive. The table should now look like this:

<u>u</u>		<u>dv</u>
x^4	$+$ ↘	$\cos x$
$4x^3$	$-$ ↘	$\sin x$
$12x^2$	$+$ ↘	$-\cos x$
$24x$	$-$ ↘	$-\sin x$
24	$+$ ↘	$\cos x$
0		$\sin x$

And now we just read the answer from the table:

$$\int x^4 \cos x dx = x^4 \sin x + 4x^3 \cos x - 12x^2 \sin x - 24x \cos x + 24 \sin x + c$$

Practice: Go to www.phschool,com and enter access code azk-1003 . Go to Chapter 6, page 346 and do # 1 – 9 (odd) and 21 along with $\int \sin^{-1} x dx$. Answers are in the back of the online book. The answer to the extra questions is $x\sin^{-1} x + \sqrt{1-x^2} + c$

Partial Fraction Decomposition

Goal: To be able to anti-derive functions of the form $\frac{\text{constant or linear function}}{\text{quadratic polynomial}}$

First let's do a Math 1 review problem.

Review: Add $\frac{2}{x+4} + \frac{7}{x}$

We need to find a common denominator [$(x)(x+4)$] and we wind up with:

$$\frac{2}{x+4} + \frac{7}{x} = \frac{2(x)}{(x+4)(x)} + \frac{7(x+4)}{x(x+4)} = \frac{9x+28}{x^2+4x}$$

With partial fraction decomposition, we start with the answer, $\frac{9x+28}{x^2+4x}$, and have to find the fractions which can be added/subtracted which will yield that as a final answer.

Decompose: $\frac{9x+28}{x^2+4x}$

Step 1: Factor the denominator: $\frac{9x+28}{x^2+4x} = \frac{9x+28}{x(x+4)}$

Step 2: Set-up one fraction for each factor of the denominator and assign each fraction a different numerator constant. $\frac{9x+28}{x(x+4)} = \frac{A}{x} + \frac{B}{x+4}$

Step 3: Solve the resulting equation by first multiplying through by the LCD.

$(x)(x+4)\left[\frac{9x+28}{x(x+4)} = \frac{A}{x} + \frac{B}{x+4}\right]$ yields $9x+28 = A(x+4) + B(x)$.

Step 4: Pick convenient values of *x* (those which make a factor on the right hand side of the equation equal to zero, and solve for *A* and *B*.

Choose *x* = -4:

$$9(-4)+28=A(-4+4)+B(-4)$$
$$-8=-4B$$
$$B=2$$

Choose *x* = 0:

$$9(0)+28=A(0+4)+B(0)$$
$$28=4A$$
$$A=7$$

Therefore: $\dfrac{9x+28}{x^2+4x}=\dfrac{7}{x}+\dfrac{2}{x+4}$

which we know is right, because that is the Math 1 review problem we started with.

Example 2: Decompose: $\dfrac{5x+2}{x^2+x}$

$$\frac{5x+2}{x^2+x}=\frac{5x+2}{x(x+1)}=\frac{A}{x}+\frac{B}{x+1}$$

$(x)(x+1)\left[\dfrac{5x+2}{x(x+1)}=\dfrac{A}{x}+\dfrac{B}{x+1}\right]$ yields $5x+2=A(x+1)+B(x)$.

Choose *x* = -1:

$$5(-1)+2=A(-1+1)+B(-1)$$
$$-3=-B$$
$$B=3$$

Choose *x* = 0:

$$5(0)+2=A(0+1)+B(0)$$
$$2=A$$

Therefore: $\dfrac{5x+2}{x^2+x}=\dfrac{2}{x}+\dfrac{3}{x+1}$

Why do we do this ? Because if think about anti-deriving the resulting partial fractions, you will see that each has an anti-derivative that is some form of the natural log function.

Example 3: Evaluate: $\displaystyle\int\frac{1}{x^2+x}\,dx$

First, decompose the integrand: $\dfrac{1}{x^2+x}=\dfrac{1}{x(x+1)}=\dfrac{A}{x}+\dfrac{B}{x+1}$

$(x)(x+1)\left[\dfrac{1}{x(x+1)}=\dfrac{A}{x}+\dfrac{B}{x+1}\right]$ yields $1=A(x+1)+B(x)$.

Choose x = -1: $1 = A(-1+1)+B(-1)$

$1 = -B$

$B = -1$

Choose x = 0: $1 = A(0+1)+B(0)$

$1 = A$

Therefore: $\frac{1}{x^2+x} = \frac{1}{x} - \frac{1}{x+1}$

So: $\int \frac{1}{x^2+x}\,dx = \int\left(\frac{1}{x} - \frac{1}{x+1}\right)dx = \ln|x| - \ln|x+1| + c = \ln\left|\frac{x}{x+1}\right| + c$ Notice how the properties of logarithms are used in the final step to simplify the answer.

Example 4: Evaluate: $\int \frac{4}{x^3-x^2-6x}\,dx$

First, decompose the integrand:

$$\frac{4}{x^3-x^2-6x} = \frac{4}{x(x^2-x-6)} = \frac{4}{x(x-3)(x+2)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+2}$$

$$(x)(x-3)(x+2)\left[\frac{4}{x(x-3)(x+2)} = \frac{A}{x} + \frac{B}{x-3} + \frac{C}{x+2}\right]$$

yields $4 = A(x-3)(x+2)+B(x)(x+2)+C(x)(x-3)$.

Choose x = 3: $4 = A(3-3)(3+2)+B(3)(3+2)+C(3)(3-3)$

$4 = 15B$

$B = \frac{4}{15}$

Choose x = -2: $4 = A(-2-3)(-2+2)+B(-2)(-2+2)+C(-2)(-2-3)$

$4 = 10C$

$C = \frac{2}{5}$

Choose x = 0: $4 = A(0-3)(0+2)+B(0)(0+2)+C(0)(0-3)$

$4 = -6A$

$A = -\frac{2}{3}$

Therefore: $\dfrac{4}{x^3 - x^2 - 6x} = -\dfrac{2}{3x} + \dfrac{4}{15(x-3)} + \dfrac{2}{5(x+2)}$

So: $\int \dfrac{4}{x^3 - x^2 - 6x}\,dx = \int\left(-\dfrac{2}{3x} + \dfrac{4}{15(x-3)} + \dfrac{2}{5(x+2)}\right)dx = -\dfrac{2}{3}\ln|x| + \dfrac{4}{15}\ln|x-3| + \dfrac{2}{5}\ln|x+2| + c$

Notice that we will **not** use the properties of logarithms to simplify the answer because of the unlike nature of the leading coefficients.

Practice: From the online book, Section 6.5, p. 369, #5 – 13 (odd) Answers are in the back of the online book.

Arc Length

Goal: To find the length of a smooth curve on a closed interval.

Given a smooth (differentiable) curve, for $y = f(x)$, the length of the curve on $[a,b]$ is given by

$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$

The very nice about this section is that we can let the calculator do all the work. ☺

****** If the curve is **not** differentiable over the entire interval, you must break the curve up into sub-intervals over which the curve is differentiable.

Example 1: Find the length of the curve $f(x) = \dfrac{x^3}{6} + \dfrac{1}{2x}$ on $\left[\dfrac{1}{2}, 2\right]$.

Step 1: Type $f(x)$ into Y_1 in your calculator.

Step 2: Let Y_2 be the derivative of Y_1 .

Step 3: Use your calculator to find the value of the integral $\int_{1/2}^{2} \sqrt{1 + [Y_2]^2}\,dx$. You should get an answer of 2.0625 or $\dfrac{33}{16}$.

Example 2: Find the perimeter of region R which is bounded by the functions $f(x) = x^2 + 5$ and $g(x) = -x^2 + 3$ on $[0,3]$. If you try this problem without looking at the procedure below, you should get a final answer of 41.494.

Solution: There are four lengths to find and to add together; the two vertical segments and each of the two curved pieces.

Left vertical: $f(0) = g(0) = 2$

Right vertical: $f(3) - g(3) = 20$

Length of $f(x) = x^2 + 5$ = $\int_0^3 \sqrt{1 + (2x)^2}\,dx = 9.747$

Length of $g(x) = -x^2 + 3$ = $\int_0^3 \sqrt{1 + (-2x)^2}\,dx = 9.747$

Add all four lengths together and you get 41.494.

Practice: From the online book, Section 7.4, p. 416, #1ac, 7ac, 9ac, 11, 27. Answers are in the back of the online book.

Population Growth

Earlier in this course we learned that populations were modeled by: $P = P_0 e^{rt}$. We obtained this by solving the **separable** differential equation $\frac{dP}{dt} = kP$

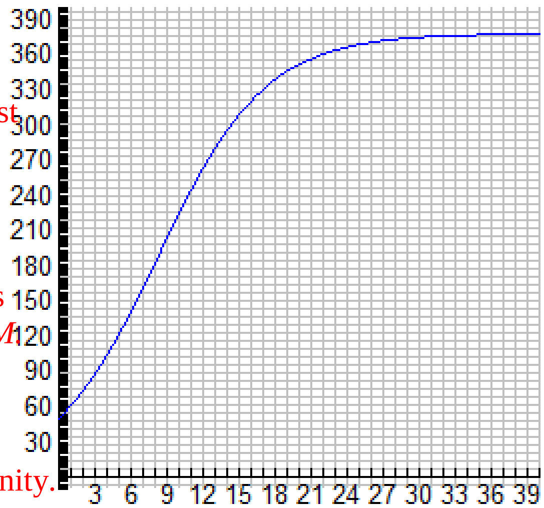
What's the problem with this model ????

Answer: This model implies that populations grow forever without bound.

Populations are actually modeled by: $P = \frac{M}{1+Ce^{-kt}}$

and their graphs resemble

This is the familiar "S" curve from biology class. The highest y-value is called the **carrying capacity** of the population. It is represented by M in the equation. You should also therefore see that the graph has a horizontal asymptote of $y = M$. And by the definition of a horizontal asymptote, M represents the limit of the population as t approaches infinity.



This is called a **logistic equation** and is obtained by solving the following **logistic differential equation:** $\frac{dP}{dt} = \frac{k}{M}P(M-P)$ or distributing gives $\frac{dP}{dt} = kP - \frac{k}{M}P^2$

Examples:

1. A certain population is modeled by the following logistic differential equation: $\frac{dP}{dt} = 0.04P - 0.0004P^2$. Find k and the carrying capacity of that population.

Answer: $k = 0.04$ and M = 100.

2. A certain population is modeled by the following logistic differential equation: $\frac{50}{P}\frac{dP}{dt} = 2 - \frac{P}{250}$. Find k and the carrying capacity of that population.

Answer: $k = 0.04$ and M = 500.

3. **<u>2004 #5 (No Calculator)</u>**

A population is modeled by a function P that satisfies the logistic differential equation:

$$\frac{dP}{dt}=\frac{P}{5}\left(1-\frac{P}{12}\right).$$

a) If $P(0)$= 3, what is $\lim_{t\to\infty} P(t)$? Mainipulate the original differential equation and you will see that M is 12.

If $P(0)$= 20, what is $\lim_{t\to\infty} P(t)$? Also 12 -- we learn that the initial population has no bearing on what the final population will be given a logistic differential equation.

b) If $P(0)$= 3, for what value of P is the population growing the fastest? When P = 6. Populations grow fasest when they are at half of their maximum.

c) A different population is modeled by a function Y that satisfies the separable differential equation $\frac{dY}{dt}=\frac{Y}{5}\left(1-\frac{t}{12}\right)$. Find $Y(t)$ if $Y(0) = 3$.

$$\frac{dY}{dt}=\frac{Y}{5}\left(1-\frac{t}{12}\right)$$

$$\frac{dY}{Y}=\frac{1}{5}\left(1-\frac{t}{12}\right)dt=\left(\frac{1}{5}-\frac{t}{60}\right)dt$$

$$\int\frac{dY}{Y}=\int\left(\frac{1}{5}-\frac{t}{60}\right)dt$$

$$\ln|Y|=\frac{1}{5}t-\frac{t^2}{120}+c$$

$$e^{\ln|Y|}=e^{\frac{1}{5}t-\frac{t^2}{120}+c}$$

$$Y=e^{\frac{1}{5}t-\frac{t^2}{120}+c}=Ce^{\frac{1}{5}t-\frac{t^2}{120}}$$

$$3=Ce^{0-0}\ \therefore C=3$$

$$Y=3e^{\frac{1}{5}t-\frac{t^2}{120}}$$

d) For the function Y found in part c), what is $\lim_{t\to\infty} Y(t)$? Substituting infinity for t we see that the final population will be zero.

<u>There is no practice for this section.</u>